Bloch martingales and conformal maps.

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Let us look closer at the distortion properties of conformal maps.

$\varphi: \mathbb{D} \to \Omega$ - conformal. Normalize: $S := \{\varphi: \mathbb{D} \to \mathbb{C}$ - conformal, $\varphi(0)=0,\ \varphi'(0)=1\}$ - compact.

Bieberbach Thm. If $\varphi(z) = z + a_2 z^2 + \dots$, $(\varphi \in S)$, then $|a_2| \le 2$.

Equality is reached iff $\varphi(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots$

lead to

Bieberbach Conjecture. $|a_n| \le n \ \forall \varphi \in S$.

Proven in 1984 by de Branges using Löwner chains.

Without normalization: if φ - conformal in $\mathbb{D}$, then $\left|\frac{\varphi''(0)}{\varphi'(0)}\right| \le 2$. Apply Möbius transformation $\tau(\zeta) = \frac{\zeta + z}{1 + \bar{z}\zeta}$, which maps z to 0, to get

$$\left| z\frac{\varphi''(z)}{\varphi'(z)} - \frac{2|z|^2}{1-|z|^2} \right| \le \frac{4|z|}{1-|z|^2}, \text{ or } |z|\left|\frac{\varphi''(z)}{\varphi'(z)}\right| \le \frac{2|z|^2}{1-|z|^2} + \frac{4|z|}{1-|z|^2} \le \frac{6|z|}{1-|z|^2}.$$

which is

$$(1-|z|^2)\left|\frac{\varphi''(z)}{\varphi'(z)}\right| \le 6 \iff (1-|z|^2)\left|(\log \varphi'(z))'\right| \le 6.$$

Why is it nice?

Def Bloch space: $\|b\|_{\mathcal{B}} := \sup_z (1-|z|^2)|b'(z)|$, b - analytic.

Meaning: b is Lipshitz from hyperbolic metric to Euclidean.

I.e. it changes by about a constant in every Whitney square.

Univalence criterium (Nehari).

$\|\log f'\|_{\mathcal{B}} \le 1 \Rightarrow f$ is conformal.

Also note that Bloch norm is Möbius invariant: $\|g \circ \tau\|_{\mathcal{B}} = \|g\|_{\mathcal{B}}$.

To prove Makarov's LIL for conformal maps, we'll prove

Law of Iterated Logarithm (LIL) for Bloch functions.

$\exists\, C > 0: \ \forall g \in \mathcal{B}$ a.e. $\zeta \in S^1$:

$$\varlimsup_{r \to 1-} \frac{|g(r\zeta)|}{\sqrt{\log\frac{1}{1-r} \log\log\log\frac{1}{1-r}}} \le C\|g\|_{\mathcal{B}}.$$

For a Bloch function g, define $G(z) := \int_0^z g(w)\,dw$.

Lemma. $G \in C_A(\bar{\mathbb{D}})$, - disk algebra satisfies Zygmund condition

$$\left|G(e^{i(\theta+t)}) + G(e^{i(\theta-t)}) - 2G(e^{i\theta})\right| \le Ct\|g\|_{\mathcal{B}}.$$

Remark. For $G \in C_A(\bar{\mathbb{D}})$, $G' \in \mathcal{B} \iff G$ is Zygmund-smooth.

Pf of Lemma. $|G(se^{i\theta}) - G(re^{i\theta})| = \left|\int_r^s g(te^{i\theta})\,dt\right| \le$

$|g(re^{i\theta})|(s-r) + \frac{\|g\|_{\mathcal{B}}}{2}\int_r^s \log\left(\frac{1+t}{1-t}\right)dt$ $\left(|g(te^{i\theta})| = \left|\int_r^t g'(\ell e^{i\theta})\,d\ell + g(re^{i\theta})\right|\right.$

$\left.\le |g(re^{i\theta})| + \int_r^t \frac{1}{1-\ell^2}\,d\ell \cdot \|g\|_{\mathcal{B}} \le |g(re^{i\theta})| + \frac{1}{2}\|g\|_{\mathcal{B}} \log\left(\frac{1+t}{1-t}\right)\right)$.

Since $\int \log\left(\frac{1+t}{1-t}\right) < \infty$, $\exists \lim G(se^{i\theta}) =: G(e^{i\theta})$

$= g(re^{i\theta}) + \int_r^1 \frac{\ldots}{1-\ell^2} d\ell \, \|g\|_B = |g(re^{i\theta})| + \frac{1}{2}\|g\|_B \log\left(\frac{1+t}{1-t}\right).$

Since $\int \log\left(\frac{1+t}{1-t}\right) < \infty$, $\exists \lim_{s \to 1-} G(se^{i\theta}) =: G(e^{i\theta})$

Moreover, for $r = 1 - \frac{t}{\pi}$,

$$|G(e^{i(\theta+t)}) + G(e^{i(\theta-t)}) - 2G(e^{i\theta})| \le \overbrace{\left|\int_r^1 g(se^{i(\theta+t)}) + g(se^{i(\theta-t)}) - 2g(se^{i\theta}) \, ds\right|}^{I} + \underbrace{\left|G(re^{i(\theta+t)}) + G(re^{i(\theta-t)}) - 2G(re^{i\theta})\right|}_{II}.$$

$$I \le 3/2 \int_r^1 \log\left(\frac{1+t}{1-t}\right) dt \, \|g\|_B + \int \left|g(se^{i(\theta+t)}) + g(se^{i(\theta-t)}) - 2g(re^{i\theta})\right|$$

$$\le \underbrace{\left|g(re^{i(\theta+t)}) + g(re^{i(\theta-t)}) - 2g(re^{i\theta})\right|}_{I'} + 6\|g\|_B \underbrace{(1-r)}_{=\frac{t}{\pi}}.$$

But $I' \le 2t\|g\|_B/(1-r^2) \le \pi\|g\|_B$.

Finally, $II \le |t|^2 \sup_{s \le s'} |g'(rs)| \le \frac{t^2 \|g\|_B}{1-r^2} \le t\pi\|g\|_B$ ■

Define now, for a dyadic interval I,

$$g_I := \lim_{r \to 1-} \frac{1}{|I|}\int_I g(re^{i\theta})\, d\theta = \lim_{r \to 1-} \frac{-i}{|I|}\int_I e^{-i\theta} \frac{dG(re^{i\theta})}{d\theta}\, d\theta =$$

(by parts) $\frac{-ie^{-ib}G(e^{ib}) + ie^{-ia}G(e^{ia})}{|I|} + \frac{1}{|I|}\int_I e^{-i\theta} G(e^{i\theta})\, d\theta$,

well-defined.

Define a function g_n by

$$g_n(\xi) = \sum_{I - \text{dyadic } n\text{-th generation}} g_I \chi_I(\xi)$$

Observe that if I, J - adjacent, then $g_{I \cup J} = \frac{1}{2}(g_I + g_J)$.

In particular, $E(g_{n+1} | \mathcal{M}_n) = g_n$, where $\mathcal{M}_n$ dyadic partition of n-th level, the IP is normalized linear measure on S^1.

So g_n is a martingale. But not just any martingale.

Def. A dyadic martingale is called Bloch if it has bounded jumps: $\exists C: \forall I, J$ - adjacent, $|f_n|_I - f_n|_J| \le C$.

Lemma $|g_n(\xi) - g((1-2^{-n})\xi)| \le C\|g\|_B$ for some absolute constant C.

Pf. By the integration by parts formula for g_I, everything reduces to comparing $\frac{1}{|I|}\int_I g((1-2^{-n})e^{i\theta})\, d\theta$ and $g((1-2^{-n})\xi)$ which is bounded, since hyperbolic diameter of $(1-2^{-n})I$ is bounded. ■

Corollary g_n is a Bloch martingale.

Pf. $|g_n|_I - g_n|_J| \le |g_I - g((1-2^{-n})\xi)| + |g_J - g((1-2^{-n})\xi)| \le 2C\|g\|_B$, where ξ is the common end of I and J. ■

Pf of LIL.

Let $g \in B$. Normalize: $\|g\|_B = 1$. Take $f_n = \mathrm{Re}\, g_n$

Then $S_n \le C n$, since $|f_{n+1} - f_n| = \left(\frac{1}{2}(f_{I_l} - f_{I_r})\right)^2 \le C$.

$\underbrace{I = I_l \cup I_r}$ – decomposition of a dyadic interval of n-th generation.

Thus, if $S < \infty$, $|f_n|$ is bounded, by Levy.
if $S = \infty$, $|f_n| \le 2\sqrt{S_n \log\log S_n} \le C\sqrt{n \log\log n}$ for large n.

Then, by Lemma,

$$\left|\operatorname{Re} g((1-2^{-n})e^{i\theta}) - f_n(\theta)\right| \le C_1, \text{ and, by Lipschitz,}$$

$$\left|\operatorname{Re} g((1-2^{-n})e^{i\theta}) - \operatorname{Re} g(re^{i\theta})\right| \le C_2, \text{ for } 1-2^{-n} \le r < 1-2^{-n+1}.$$

Thus

$$|\operatorname{Re} g(re^{i\theta})| \le C_3\sqrt{\log\frac{1}{1-r} \log\log\log\frac{1}{1-r}} \text{ a.s.}$$

Now do the same with $\operatorname{Im} g(re^{i\theta})$.

Remark. All of this is sharp: every real Bloch martingale is generated by the real part of a Bloch function. When the Bloch norm is small, Nehari's criterion implies that $b = \log\varphi'$ for some φ – conformal. This allows us to prove that the lower bound in LIL can be achieved, with different C.